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Tensor Viscosity Method for Convection in Numerical Fluid Dynamics*

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A new method, called the tensor viscosity method, is described for differencing the convective terms in multidimensional numerical fluid dynamics. The method is the proper generalization to two or three dimensions of interpolated donor cell differencing in one dimension, and is designed to achieve numerical stability with minimal numerical damping. It is a single-step method that is distinguished by simplicity and ease of implementation, even in the case of an arbitrary non-rectangular mesh. It should therefore be useful in finite-element as well as finite-difference formulations.

I. INTRODUCTION

In numerical fluid dynamics, the manner in which the convective terms are spatially differenced requires special attention. As is well known, when these terms are evaluated explicitly (i.e., at the previous time level) centered spatial differencing is unconditionally unstable [1, 2]. This instability can be traced to a destabilizing truncation error proportional to the square of the fluid velocity [3]. Various methods for circumventing this problem have been used over the years. The simplest and least accurate procedure is to simply add artificial diffusive terms to the equations, with their coefficients sufficiently large to ensure numerical stability. A better but still inaccurate procedure is to use donor cell or upwind differencing [1, 2, 4]. These methods are too inaccurate for many problems because of the unphysical diffusion or smearing that attends their use.

A much better procedure is the use of a weighted average of centered and donor cell differencing. The fraction of donor cell differencing needed for stability is approximately the fraction of a cell width traversed by the fluid in one timestep. This fraction is ordinarily considerably less than unity, and the unphysical diffusion is consequently greatly reduced. However, if the fluid velocity varies greatly throughout the region of computation, the use of a single weighting factor everywhere will still result in unnecessarily large numerical smearing in regions where the velocity is relatively small. The obvious way to avoid this disadvantage is to locally set the weighting factor

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in terms of the local fluid velocity. This procedure is referred to as interpolated donor cell differencing [5], and in one dimension it is equivalent to the local cancellation of the destabilizing truncation error mentioned above.

Unfortunately, the naive extension of interpolated donor cell differencing to two dimensions is unstable; this follows from its equivalence to the scheme given in Eq. (39) of Ref. [6], or Eq. (5) of Ref. [7]. The difficulty is due to the fact that the destabilizing truncation error in two dimensions contains an xy cross term for which interpolated donor cell differencing does not compensate. This cross term is implicitly cancelled by the well-known second-order two-step methods, such as the Lax–Wendroff and MacCormack methods [1, 2]. These methods, however, also introduce into the difference scheme various higher-order cross terms whose significance is unclear. A simpler way around the difficulty is to compensate directly, in a single step, for the complete two-dimensional form of the destabilizing truncation error, including the cross term. The compensating term has the form of a viscous term in which the viscosity coefficient is a tensor rather than a scalar; we therefore refer to this method as the tensor viscosity (TV) method. It is the proper generalization to two (or three) dimensions of interpolated donor cell differencing in one dimension.

The TV method is simple and easy to implement, even in an arbitrary nonrectangular mesh or in orthogonal curvilinear coordinates. This is a consequence of the fact that the tensor viscosity term is a spatial difference approximation to a differential term of tensor-invariant form, which can therefore be spatially differenced in the same manner as the other terms in the equations. This feature should make the method useful in finite-element as well as finite-difference contexts. The TV method has the further advantage that its form is the same in three dimensions as in two dimensions.

We emphasize that the present method is independent of and unrelated to the various earlier numerical methods with which the term "tensor viscosity" has been associated [8]. These methods were used primarily for shock smearing [9], usually within a Lagrangian context. The present method is simply a way of stabilizing Eulerian convection, and has nothing to do with shock smearing. The method is not needed and should not be used in Lagrangian calculations, where there is no convection with respect to the finite difference mesh.

II. Derivation of the Tensor Viscosity Term

Consider the continuity equation in either two or three dimensions,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{1}$$

where ρ is the mass density and $\mathbf{u}$ is the fluid velocity vector. As is well known, forward-time centered-space (FTCS) differencing of Eq. (1) is unconditionally unstable [1, 2]. This instability arises because the term $\nabla \cdot (\rho \mathbf{u})$ is evaluated at an earlier time than the term $\partial \rho / \partial t$; if both terms are evaluated at the same time level (at least to second

order in Δt) a stable scheme results [1, 2]. Thus it is clear that the FT part of the FTCS scheme, and not the CS part, is the source of the trouble. This suggests that further insight can be gained by considering the temporal differencing separately from the spatial differencing. Accordingly, we leave the spatial variables continuous and discretize only the time to obtain the FT temporal difference approximation to Eq. (1),

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} + \nabla \cdot (\rho \mathbf{u})^n = 0. \tag{2}$$

Here Δt is the time increment and superscript n is the time level, so that $t = n\,\Delta t$.

Let us now examine the truncation errors of Eq. (2) [3]. Using Eq. (1), we find that

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} = \left(\frac{\partial \rho}{\partial t}\right)^n + \frac{1}{2}\Delta t\, \nabla \cdot \mathbf{F}^n + \mathcal{O}(\Delta t^2), \tag{3}$$

where

$$\mathbf{F} = \mathbf{u}\mathbf{u} \cdot \nabla \rho + \rho \mathbf{u} \nabla \cdot \mathbf{u} - \rho\, \partial \mathbf{u}/\partial t. \tag{4}$$

The FT difference Eq. (2) is therefore equivalent to the equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = -\frac{1}{2}\Delta t\, \nabla \cdot \mathbf{F} + \mathcal{O}(\Delta t^2), \tag{5}$$

evaluated at time level n. Equation (5) may be considered to be the differential equation that is *effectively* solved by the use of Eq. (2). Notice that $1/2\ \Delta t\ \mathbf{F}$ has the significance of an additional mass flux.

The first term in $\mathbf{F}$ is the one of crucial importance. This term gives rise to an artificial diffusional mass flux in which the diffusivity is replaced by the dyadic $-1/2\ \Delta t\ \mathbf{uu}$. The negative sign in this expression implies that Eq. (5) will exhibit unbounded instabilities [3]. The other terms in $\mathbf{F}$ do not appear to be susceptible to similar interpretations. Moreover, these terms are proportional to derivatives of $\mathbf{u}$, while the instability of present concern occurs even when $\mathbf{u}$ is independent of position and time [1, 2]. One therefore concludes that these terms are not essential to the basic FT instability.

Having thus identified the first term in $\mathbf{F}$ as the cause of the instability, it is a simple matter to devise an alternative numerical scheme which explicitly compensates for this term. To do so, we simply incorporate the troublesome term into the difference approximation to $(\partial \rho/\partial t)^n$, in the manner dictated by Eqs. (3) and (4). That is, we approximate $(\partial \rho/\partial t)^n$ by the difference expression

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} - \frac{1}{2}\Delta t\, \nabla \cdot (\mathbf{u}\mathbf{u} \cdot \nabla \rho)^n. \tag{6}$$

This is a first-order approximation overall, but is of second order in the effect of essential importance. The difference scheme then becomes, instead of Eq. (2),

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} + \nabla \cdot (\rho \mathbf{u})^n = \nabla \cdot (\mathbf{T} \cdot \nabla \rho)^n, \tag{7}$$

where we have introduced the "tensor viscosity" $\mathbf{T}$, defined by

$$\mathbf{T} = \tfrac{1}{2} \Delta t \, \mathbf{u}\mathbf{u}. \tag{8}$$

Clearly $\mathbf{T}$ is a symmetric tensor. Note that, in spite of the appearance of Eq. (7), the term involving $\mathbf{T}$ is not to be thought of as an additional artificial term, but rather as part of the difference approximation to $(\partial \rho / \partial t)^n$.

It is clear that the preceding considerations apply equally well to the convective transport of an arbitrary quantity Q satisfying the equation

$$\frac{\partial Q}{\partial t} + \nabla \cdot (Q\mathbf{u}) = S_Q, \tag{9}$$

where Q is a volumetric density (quantity per unit volume) and S_Q represents the non-convective contributions to $\partial Q / \partial t$. The additional truncation errors that arise when $S_Q \neq 0$ are again not of importance to the basic FT instability and hence there is no need to compensate for them. We therefore temporally difference Eq. (9) as

$$\frac{Q^{n+1} - Q^n}{\Delta t} + \nabla \cdot (Q\mathbf{u})^n = S_Q + \nabla \cdot (\mathbf{T} \cdot \nabla Q)^n. \tag{10}$$

The time level at which S_Q is evaluated is not shown because it is immaterial for present purposes; it might be either n or $n + 1$. The conservation equations of fluid dynamics are all of the form of Eq. (9), and hence can all be temporally differenced according to Eq. (10). Note that $\mathbf{T}$ is the same for all quantities Q.

To complete the difference scheme, it is merely necessary to replace the spatial derivatives in Eq. (10) by spatial differences formed with reference to the finite-difference mesh. This will ordinarily be done by the simplest symmetrical or centered procedure, leading to spatial difference expressions that agree with their differential counterparts to second order in a uniform rectangular mesh and to first order otherwise.

III. Discussion

Although the tensor viscosity term arises from the time derivative, it may alternatively be regarded as part of the convective flux. This is permissible because the difference expression

$$(Q\mathbf{u})^n - (\mathbf{T} \cdot \nabla Q)^n \tag{11}$$

is a first-order difference approximation to $Q\mathbf{u}$. Indeed, this interpretation is essential if one wants the difference equations to preserve the integral balances implied by the differential system (for example, the integral of the mass flux over a closed surface vanishes in steady state). Since the tensor viscosity term has been derived in conservation (i.e., divergence) form, it lends itself naturally to the construction of conservative difference schemes based on this interpretation.

It is instructive, especially for purposes of comparison, to write out the explicit matrix form of $\mathbf{T}$ for the case of two-dimensional rectangular Cartesian coordinates (x, y):

$$\mathbf{T} = \tfrac{1}{2}\Delta t \begin{pmatrix} u^2 & uv \\ uv & v^2 \end{pmatrix}, \tag{12}$$

where u and v are respectively the x- and y-components of $\mathbf{u}$. It is also possible to express the convective fluxes in both donor cell and interpolated donor cell differencing as difference approximations to expressions of the form (11). The donor cell form of $\mathbf{T}$ is

$$\mathbf{T} = \tfrac{1}{2}\Delta t \begin{pmatrix} \dfrac{|u|\Delta x}{\Delta t} & 0 \\ 0 & \dfrac{|v|\Delta y}{\Delta t} \end{pmatrix}, \tag{13}$$

where Δx and Δy are the spatial increments in the x and y directions. The interpolated donor cell form of $\mathbf{T}$ is

$$\mathbf{T} = \tfrac{1}{2}\Delta t \begin{pmatrix} u^2 & 0 \\ 0 & v^2 \end{pmatrix}. \tag{14}$$

Note that interpolated donor cell differencing is equivalent to replacing the off-diagonal elements in Eq. (12) by zero. That this is not a good idea is suggested by the fact that it destroys the tensor invariance; the interpolated donor cell $\mathbf{T}$, given by Eq. (14), clearly does not transform as a tensor. It was this observation that originally suggested to us the use of Eq. (12) instead. The fact that interpolated donor cell differencing in two dimensions is inherently unstable [6] is evidence that tensor invariance is not merely of aesthetic significance. The tensor invariance is also very convenient for constructing spatial differences; since the tensor viscosity term $\nabla \cdot (\mathbf{T} \cdot \nabla Q)$ is already a tensor-invariant differential term, it may be spatially differenced in exactly the same manner as the other terms in the equations. This is especially useful in an arbitrary nonrectangular mesh, or in finite-element formulations, where the appropriate forms for donor cell or interpolated donor cell fluxes are far from obvious. Indeed, one of the principal advantages of the TV method is that its implementation in an arbitrary mesh, or in generalized coordinates, is straightforward.

It must be emphasized, however, that the tensor invariance of the differential term $\nabla \cdot (\mathbf{T} \cdot \nabla Q)$ does not carry over to the difference scheme. The spatial differencing introduces anisotropy because the mesh has preferred directions. Thus, for example,

an axially symmetric problem in a rectangular mesh will exhibit deviations from axial symmetry (although these deviations are generally somewhat less with the TV method than with donor cell differencing). This problem is common to all difference schemes when the mesh does not share the symmetry of the problem.

The fact that interpolated donor cell differencing in two dimensions is unstable has sometimes been overlooked, possibly because in practice the instability is mitigated by the presence of physical or numerical dissipative effects, and in addition it is localized in regions where the flow is primarily along a cell diagonal. Nevertheless, the scheme is unstable and hence unacceptable. Stability is restored by introducing the missing off-diagonal elements to obtain the TV method. A Fourier analysis of the TV method for the canonical test case (linear advection by a uniform time-independent velocity field in a uniform two-dimensional rectangular mesh) leads to the stability restriction

$$\Delta t \leqslant (\beta_1 + \beta_2)^{-2}(\beta_1^2 + \beta_2^2)^{1/2}, \tag{15}$$

where $\beta_1 = |u|/\Delta x$ and $\beta_2 = |v|/\Delta y$. When $|v| = 0$ this reduces to $\Delta t \leqslant \Delta x/|u|$, the usual one-dimensional interpolated donor cell restriction.

Stability is also restored by the use of full donor cell differencing, whose corresponding stability restriction is [2]

$$\Delta t \leqslant (\beta_1 + \beta_2)^{-1}. \tag{16}$$

This is always less restrictive than the TV condition (15), but never by more than a factor of $2^{1/2}$. However, this stability is achieved by introducing additional numerical damping and by doing further violence to the tensor invariance. Instead of increasing the off-diagonal elements in the appropriate manner, donor cell differencing leaves them zero and greatly increases the diagonal elements instead. Indeed, these elements exceed the corresponding TV elements by factors of $\Delta x/|u| \Delta t$ and $\Delta y/|v| \Delta t$, which can be very large. Because of the excessive numerical smearing caused by these large diagonal elements, donor cell differencing is too inaccurate for many purposes. Although this is now widely recognized, donor cell differencing has remained in common use because of the absence of a more satisfactory method of comparable simplicity, such as the TV method.

The artificial viscosity of a numerical scheme is of primary importance; it is usually defined with reference to the diffusional truncation errors of the difference scheme in the canonical test case mentioned above. Since the TV method cancels these very errors, it has no artificial viscosity in this sense, in contrast to donor cell differencing. However, this applies only to the transient and not to the steady state [10]. In the steady state the tensor viscosity term can no longer be regarded as part of the time derivative, and therefore becomes an artificial viscous term in the equations. The magnitude of this term can be reduced by reducing Δt, which will change the steady-state solution, just as it does in the various second-order methods [1, 2].

Although the truncation error on which the TV method is based is the critical one, the difference scheme of course contains other truncation errors. These errors may be

destabilizing, dispersive, or dissipative, and in general their magnitudes depend on local flow conditions. In some problems these errors can lead to irregular and/or marginally unstable solutions. To allow for this possibility it is advisable to insert a variable factor α into the right-hand side of Eq. (8). The nominal value $\alpha = 1$ corresponds to exact cancellation of the critical convective truncation error. Larger values of α can be used to provide additional smoothing for cases in which smooth and stable solutions do not obtain with $\alpha = 1$. The introduction of α does not alter the formal order of accuracy of the TV scheme, but it does affect the absolute accuracy; thus α should be kept as close to unity as possible.

We have obtained the form of the TV term by an argument based on Taylor series expansions. Such expansions are legitimate only when the field in question is sufficiently smooth. This is not always the case in inviscid flows, where the field variables or their low-order derivatives may be discontinuous. For many purposes, it is sufficient to represent such discontinuities numerically as relatively rapid but continuous transitions (e.g.. shock smearing [9]), in which case the TV method remains applicable (provided the zoning is fine enough). If it is necessary to represent discontinuities as such, then special techniques are required.

IV. Numerical Example

As a simple example we consider the solution of Eq. (1) in two dimensions, in the square region $0 \leqslant x, y \leqslant X$. The velocity field is uniform with components $u = v = U_0$; thus the flow is purely diagonal. The initial density in the region is uniform with the value ρ_1, and the boundary conditions are that $\rho = \rho_2$ on the inflow boundaries $x = 0$ and $y = 0$. The exact solution of this problem is an L-shaped step-function density wave, with density jump $\rho_1 - \rho_2$, propagating diagonally into the region with speed $2^{1/2}U_0$.

Finite-difference numerical solutions to this problem were generated in a uniform rectangular xy mesh using both donor cell differencing and the TV method (with $\alpha = 1$). In both cases the problem parameters were

$$\begin{aligned} \rho_1 &= 1.0, & \rho_2 &= 2.0, \\ U_0 &= 1.0, & \Delta t &= 0.2, \\ \Delta x &= \Delta y = 1.0, & X &= 18, \end{aligned}$$

and simple extrapolation outflow boundary conditions were used at $x = X$ and $y = X$. Density contours at $t = 4.0$ for the two cases are shown in Fig. 1. The additional numerical smearing of the donor cell method is readily apparent. The density contours at the later time $t = 8.0$ are shown in Fig. 2. Between $t = 4.0$ and $t = 8.0$ the width of the front (as measured by the distance between the H and L contour lines) has increased about 16% in the TV case, while in the donor cell case it has increased about 41%. The sharp corner of the front is also being rounded much more severely by donor cell differencing than by the TV method.

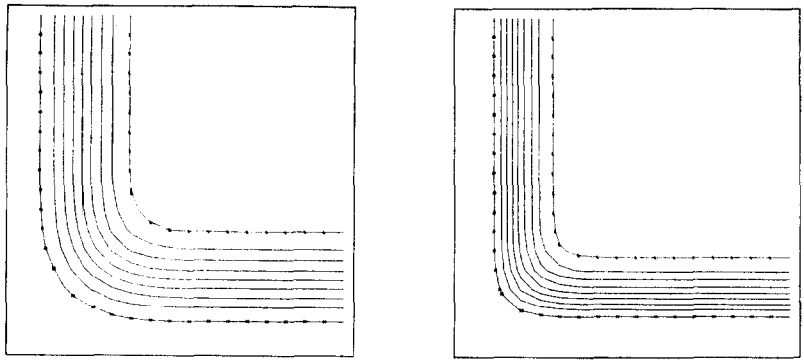

FIG. 1. Density contours at $t = 4.0$ for donor cell (left) and tensor viscosity (right) cases. Contours are equally spaced between the L value ($\rho_L = 1.1$) and the H value ($\rho_H = 1.9$).

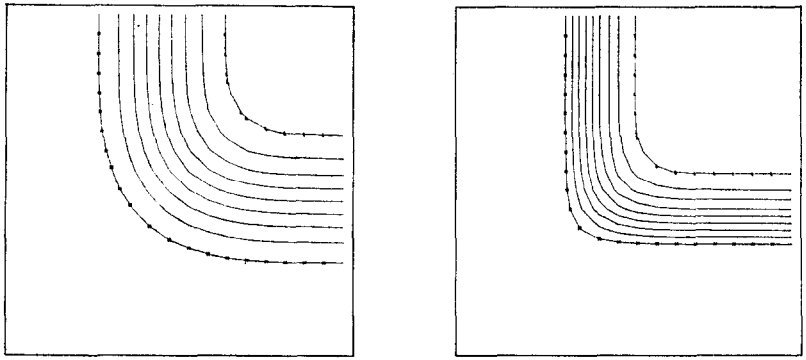

FIG. 2. Density contours at $t = 8.0$ for donor cell (left) and tensor viscosity (right) cases. Contours are equally spaced between the L value ($\rho_L = 1.1$) and the H value ($\rho_H = 1.9$).

We are currently using the TV method in an Eulerian fluid dynamics code called APACHE [11], and have found it to give good results in a wide variety of test problems. In problems where large spatial gradients exist, it is sometimes necessary to use a value of α substantially greater than unity. Rarely, however, is a value so large as $\Delta x/|u|_{\max} \Delta t$ needed, and even then the associated damping is comparable to that of donor cell differencing only where the velocity is largest, and not in the rest of the mesh.

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